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E_7 and the tripartite entanglement of seven qubits

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ABSTRACT

In quantum information theory, it is well known that the tripartite entanglement of three qubits is described by the group $[SL(2,C)]^3$ and that the entanglement measure is given by Cayley's hyperdeterminant. This has provided an analogy with certain $N = 2$ supersymmetric black holes in string theory, whose entropy is also given by the hyperdeterminant. In this paper, we extend the analogy to $N = 8$. We propose that a particular tripartite entanglement of seven qubits, encoded in the Fano plane, is described by the exceptional group $E_7(C)$ and that the entanglement measure is given by Cartan's quartic E_7 invariant.

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1 Cayley's hyperdeterminant, black holes and qubits

In 1845 Cayley [1] generalized the determinant of a 2×2 matrix a_{AB} to the *hyperdeterminant* of a $2 \times 2 \times 2$ *hypermatrix* a_{ABD}

$$
\begin{aligned}
\text{Det } a = & -\frac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3}a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4} \\
= & \; a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{100}^2a_{011}^2 \\
& -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\
& +a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\
& +a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\
& +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111})
\end{aligned} \tag{1.1}
$$

The hyperdeterminant vanishes iff the following system of equations in six unknowns p^A, q^B, r^D has a nontrivial solution, not allowing any of the pairs to be both zero:

$$
\begin{aligned}
a_{ABD}p^Aq^B &= 0 \\
a_{ABD}p^Ar^D &= 0 \\
a_{ABD}q^Br^D &= 0
\end{aligned} \tag{1.2}
$$

For our purposes, the important properties of the hyperdeterminant are that it is a quartic invariant under $[SL(2)]^3$ and under a triality that interchanges A, B and D. These properties are valid whether the a_{ABD} are complex, real or integer.

The hyperdeterminant makes it appearance in quantum information theory [2]. Let the three qubit system ABD (Alice, Bob and Daisy) be in a pure state $|\Psi\rangle$, and let the components of $|\Psi\rangle$ in the standard basis be a_{ABD}:

$$|\Psi\rangle = a_{ABD}|ABD\rangle \tag{1.3}$$

Then the three way entanglement of the three qubits A, B and D is given by the *3-tangle* [3]

$$\tau_3(ABD) = 4|\text{Det } a_{ABD}|. \tag{1.4}$$

However, one of us recently found another physical application of this hyperdeterminant [4] by associating the eight components of a_{ABD} with the four electric and four magnetic charges of the STU black hole in four-dimensional string theory [5] and showing that its entropy [6] is given by

$$S = \pi\sqrt{|\text{Det } a_{ABD}|}. \tag{1.5}$$

As far as we can tell [4], the appearance of the Cayley hyperdeterminant in these two different contexts of stringy black hole entropy (where the a_{ABD} are integers and the symmetry is $[SL(2,Z)]^3$) and three-qubit quantum entanglement (where the a_{ABD} are complex numbers and the symmetry is $[SL(2,C]^3$) is a purely mathematical coincidence. Nevertheless,

it has already provided fascinating new insights [7, 8] into the connections between strings, black holes, and quantum information[3].

The black holes described by Cayley's hyperdeterminant are those of $N = 2$ supergravity coupled to three vector multiplets, where the symmetry is $[SL(2, Z)]^3$. One might therefore ask whether the black hole/information theory correspondence could be generalized. There are three generalizations we might consider:

1) $N = 2$ supergravity coupled to l vector multiplets where the symmetry is $SL(2, Z) \times SO(l - 1, 2, Z)$ and the black holes carry charges belonging to the $(2, l + 1)$ representation ($l + 1$ electric plus $l + 1$ magnetic).

2) $N = 4$ supergravity coupled to m vector multiplets where the symmetry is $SL(2, Z) \times SO(6, m, Z)$ where the black holes carry charges belonging to the $(2, 6 + m)$ representation ($m + 6$ electric plus $m + 6$ magnetic).

3) $N = 8$ supergravity where the symmetry is the non-compact exceptional group $E_{7(7)}(Z)$ and the black holes carry charges belonging to the fundamental 56-dimensional representation (28 electric plus 28 magnetic).

In all three case there exit quartic invariants akin to Cayley's hyperdeterminant whose square root yields the corresponding black hole entropy. If there is to be a quantum information theoretic interpretation, however, it cannot just be random entanglement of more qubits, because the general n qubit entanglement is described by the group $[SL(2, C)]^n$, which, even after replacing Z by C, differs from the above symmetries (except when $n = 3$, which correspond to case (1) above with $l = 3$, the case we already know.).

In this paper we focus on the $N = 8$ case and, noting that

$$E_{7(7)}(Z) \supset [SL(2, Z)]^7 \tag{1.6}$$

and

$$E_7(C) \supset [SL(2, C)]^7, \tag{1.7}$$

we show that the corresponding system in quantum information theory is that of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George). However, the larger symmetry requires that they undergo at most tripartite entanglement of a very specific kind. The entanglement measure will be given by the quartic Cartan $E_7(C)$ invariant [11, 12, 13, 14]. The entanglement may be represented by the Fano plane [10] which also provides the multiplication table of the octonions[4].

The $N = 4$ case (2) with $m = 6$ is a subsector of the $N = 8$ case (3) and can be shown also to correspond to particular tripartite entanglement of seven qubits. The familiar $N = 2$ case with $l = 3$ is also a subsector, but with three qubits.

[3]A third application [9], not considered in this paper, is the Nambu-Goto string whose action is also given by $\sqrt{|\text{Det } a_{ABD}|}$ in spacetime signature $(2, 2)$.

[4]We are grateful to Murat Gunaydin for pointing out the connection between our entanglement diagram, Figure 1, and the multiplication table of the octonions. This result was announced by one of us (MJD) at the Supergravity at 30 Conference, Paris, 19/10/06.

2 Decomposition of $E_{7(7)}$

Consider the decomposition of the fundamental 56-dimensional representation of $E_{7(7)}$ under its maximal subgroup

$$\begin{aligned} E_{7(7)} &\supset SL(2)_A \times SO(6,6) \\ 56 &\to (2,12) + (1,32) \end{aligned} \tag{2.1}$$

Further decomposing $SO(6,6)$,

$$\begin{aligned} SL(2)_A \times SO(6,6) &\supset SL(2)_A \times SL(2)_B \times SL(2)_D \times SO(4,4) \\ (2,12) + (1,32) &\to (2,2,2,1) \\ &+ (2,1,1,8_v) + (1,2,1,8_s) + (1,1,2,8_c) \end{aligned} \tag{2.2}$$

Further decomposing $SO(4,4)$,

$$\begin{aligned} &SL(2)_A \times SL(2)_B \times SL(2)_D \times SO(4,4) \supset SL(2)_A \times SL(2)_B \times SL(2)_D \\ &\times SO(2,2) \times SO(2,2) \\ &(2,2,2,1) + (2,1,1,8_v) + (1,2,1,8_s) + (1,1,2,8_c) \to \\ &(2,2,2,1,1) + (2,1,1,4,1) + (2,1,1,1,4) \\ &+(1,2,1,2,2) + (1,2,1,2,2) + (1,1,2,2,2) + (1,1,2,2,2) \end{aligned} \tag{2.3}$$

Finally, further decomposing each $SO(2,2)$

$$\begin{aligned} &SL(2)_A \times SL(2)_B \times SL(2)_D \times SO(2,2) \times SO(2,2) \supset \\ &SL(2)_A \times SL(2)_B \times SL(2)_D \times SL(2)_C \times SL(2)_G \times SL(2)_F \times SL(2)_E \\ &(2,2,2,1,1) + (2,1,1,4,1) + (2,1,1,1,4) \\ &+(1,2,1,2,2) + (1,2,1,2,2) + (1,1,2,2,2) + (1,1,2,2,2) \to \\ &(2,2,2,1,1,1,1) + (2,1,1,2,2,1,1) + (2,1,1,1,1,2,2)+ \\ &(1,2,1,2,1,1,2) + (1,2,1,1,2,2,1) + (1,1,2,2,1,2,1) + (1,1,2,1,2,1,2) \end{aligned}$$

In summary,

$$E_{7(7)} \supset SL(2)_A \times SL(2)_B \times SL(2)_C \times SL(2)_D \times SL(2)_E \times SL(2)_F \times SL(2)_G \tag{2.4}$$

and the 56 decomposes as

$$\begin{aligned} &56 \to \\ &(2,2,1,2,1,1,1) \\ &+(1,2,2,1,2,1,1) \\ &+(1,1,2,2,1,2,1) \\ &+(1,1,1,2,2,1,2) \\ &+(2,1,1,1,2,2,1) \\ &+(1,2,1,1,1,2,2) \\ &+(2,1,2,1,1,1,2) \end{aligned} \tag{2.5}$$

An analogous decomposition holds for

$$E_7(C) \supset [SL(2,C)]^7. \tag{2.6}$$

3 Tripartite entanglement of 7 qubits

We have seen that in the case of three qubits, the tripartite entanglement is described by $[SL(2,C)]^3$ and that the entanglement measure is given by Cayley's hyperdeterminant. Now we consider seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George) but where Alice has tripartite entanglement not only with Bob/Daisy but also with Emma/Fred and also with George/Charlie, and similarly for the other six individuals. So, in fact, each person has tripartite entanglement with each of the remaining three couples:

$$
\begin{aligned}
&|\Psi\rangle = \\
&a_{ABD}|ABD\rangle \\
&+b_{BCE}|BCE\rangle \\
&+c_{CDF}|CDF\rangle \\
&+d_{DEG}|DEG\rangle \\
&+e_{EFA}|EFA\rangle \\
&+f_{FGB}|FGB\rangle \\
&+g_{GAC}|GAC\rangle
\end{aligned}
\tag{3.1}
$$

Note that:
1) Any pair of states has an individual in common
2) Each individual is excluded from four out of the seven states
3) Two given individuals are excluded from two out of the seven states
4) Three given individuals are never excluded

The entanglement may be represented by a heptagon with vertices A,B,C,D,E,F,G and seven triangles ABD, BCE, CDF, DEG, EFA, FGB, and GAC. See Figure 1. Alternatively, we can use the Fano plane. See Figure 2. The Fano plane corresponds to the multiplication table of the octonions as may be seen from the description of the state $|\Psi\rangle$ given in Table 1.

Each of the seven states transforms as a $(2,2,2)$ under three of the $SL(2)$'s and are singlets under the remaining four. Note that from (2.2) we see that the A-B-D triality of section 1 is linked with the $8_v - 8_s - 8_c$ triality of the $SO(4,4)$. Individually, therefore, the tripartite entanglement of each of the seven states is given by Cayley's hyperdeterminant. Taken together however, we see from (2.5) that they transform as a complex 56 of $E_7(C)$. Their tripartite entanglement must be is given by an expression that is quartic in the coefficients a,b,c,d,e,f,g and invariant under $E_7(C)$. The unique possibility is the Cartan invariant I_4, and so the 3-tangle is given by

$$\tau_3(ABCDEFG) = 4|I_4| \tag{3.2}$$

If the wave-function (3.1) is normalized, then $0 \leq \tau_3(ABCDEFG) \leq 1$.

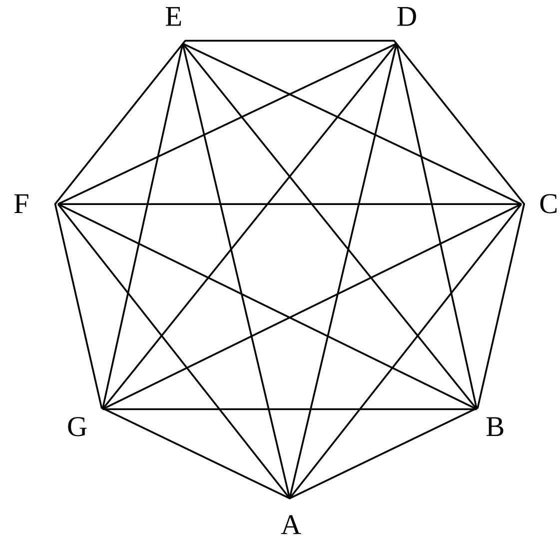

Figure 1: The E_7 entanglement diagram. Each of the seven vertices A,B,C,D,E,F,G represents a qubit and each of the seven triangles ABD, BCE, CDF, DEG, EFA, FGB, GAC describes a tripartite entanglement.

4 Cartan's $E_{7(7)}$ invariant

The Cremmer-Julia [12] form of the Cartan $E_{7(7)}$ invariant may be written as

$$I_4 \;=\; \mathrm{T}r(Z\bar{Z})^2 - \tfrac{1}{4}(\mathrm{T}r\, Z\bar{Z})^2 + 4(\mathrm{Pf}\ Z + \mathrm{Pf}\ \bar{Z}\,)\ , \tag{4.1}$$

and the Cartan form [11] may be written as

$$I_4 \;=\; -\mathrm{T}r(\,x\,y)^2 + \tfrac{1}{4}(\mathrm{T}r\,x\,y)^2 - 4(\mathrm{Pf}\ x + \mathrm{Pf}\ y\,)\ . \tag{4.2}$$

Here

$$Z_{AB} = -\frac{1}{4\sqrt{2}}(x^{ab} + iy_{ab})(\Gamma^{ab})_{AB} \tag{4.3}$$

and

$$x^{ab} + iy_{ab} = -\frac{\sqrt{2}}{4} Z_{AB}(\Gamma^{AB})_{ab} \tag{4.4}$$

The matrices of the $SO(8)$ algebra are $(\Gamma^{ab})_{AB}$ where $(a\,b)$ are the 8 vector indices and (A, B) are the 8 spinor indices. The $(\Gamma^{ab})_{AB}$ matrices can be considered also as $(\Gamma^{AB})_{ab}$ matrices due to equivalence of the vector and spinor representations of the $SO(8)$ Lie algebra. The exact relation between the Cartan invariant in (4.2) and Cremmer-Julia invariant [12] in (4.1) was established in [15, 16]. The quartic invariant I_4 of $E_{7(7)}$ is also related to the octonionic Jordan algebra J_3^O [14].

In the stringy black hole context, Z_{AB} is the central charge matrix and (x, y) are the quantized charges of the black hole (28 electric and 28 magnetic). The relation between the entropy of stringy black holes and the Cartan-Cremmer-Julia $E_{7(7)}$ invariant was established

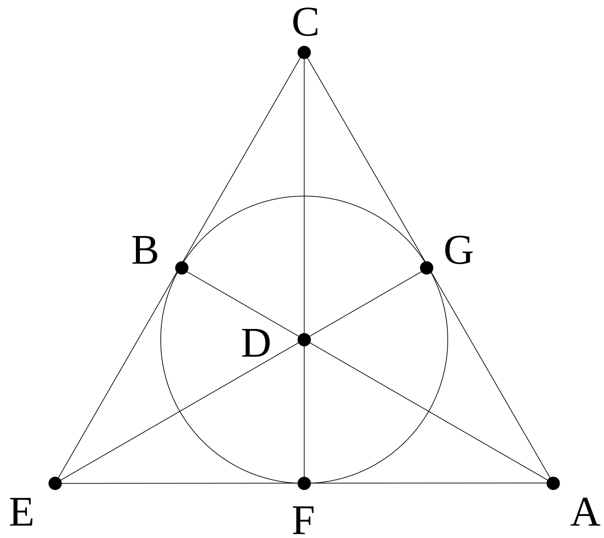

Figure 2: The Fano plane has seven points, representing the seven qubits, and seven lines (the circle counts as a line) with three points on every line, representing the tripartite entanglement, and three lines through every point.

in [13]. The central charge matrix Z_{AB} can be brought to the canonical basis for the skew-symmetric matrix using an $SU(8)$ transformation:

$$Z_{ab} = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.5}$$

where $z_i = \rho_i e^{i\varphi_i}$ are complex. In this way the number of entries is reduced from 56 to 8. In a systematic treatment in [17], the meaning of these parameters was clarified. From 4 complex values of $z_i = \rho_i e^{i\varphi_i}$ one can remove 3 phases by an $SU(8)$ rotation, but the overall phase cannot be removed; it is related to an extra parameter in the class of black hole solutions [18, 19]. In this basis, the quartic invariant takes the form [13]

$$\begin{aligned} I_4 &= \sum_i |z_i|^4 - 2\sum_{i<j} |z_i|^2 |z_j|^2 + 4(z_1 z_2 z_3 z_4 + \bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4) \\ &= (\rho_1+\rho_2+\rho_3+\rho_4)(\rho_1+\rho_2-\rho_3-\rho_4)(\rho_1-\rho_2+\rho_3-\rho_4)(\rho_1-\rho_2-\rho_3+\rho_4) \\ &\quad +8\rho_1\rho_2\rho_3\rho_4(\cos\varphi - 1) \end{aligned} \tag{4.6}$$

Therefore a 5-parameter solution is called a generating solution for other black holes in N=8 supergravity/M-theory. The expression for their entropy is always given by

$$S = \pi\sqrt{|I_4|} \tag{4.7}$$

for some subset of 5 of the 8 parameters mentioned above. Recently a new class of solutions was discovered, describing black rings. The maximal number of parameters for the known

	A	B	C	D	E	F	G
A		D	G	-B	F	-E	-C
B	-D		E	A	-C	G	-F
C	-G	-E		F	B	-D	A
D	B	-A	-F		G	C	-E
E	-F	C	-B	-G		A	D
F	E	-G	D	-C	-A		B
G	C	F	-A	E	-D	-B	

Table 1: The entanglement of the state $|\Psi\rangle$ coincides with the multiplication table of the octonions.

solutions is 7. The entropy of black ring solutions found so far was identified in [20, 21] with the expression (4.7) for a subset of 7 out of 8 parameters mentioned above.

Kallosh and Linde have shown that I_4 depending on 4 complex eigenvalues can be represented as Cayley's hyperdeterminant of a hypermatrix a_{ABD}. To see this, we that in x, y basis only the $SO(8)$ symmetry is manifest, which means that every term in (4.2) is invariant only under $SO(8)$ symmetry. However, it was proved in [11] and [12] that the sum of all terms in (4.2) is invariant under the full $SU(8)$ symmetry, which acts as follows

$$\delta(x^{ab} \pm iy_{ab}) = (2\Lambda^{[a}{}_{[c}\delta^{b]}{}_{d]} \pm i\Sigma_{abcd})(x^{cd} \mp iy_{cd}) \ . \tag{4.8}$$

The total number of parameters is 63, where 28 are from the manifest $SO(8)$ and 35 from the antisymmetric self-dual $\Sigma_{abcd} = {}^*\Sigma^{abcd}$. Thus one can use the $SU(8)$ transformation of the complex matrix $x^{ab} + iy_{ab}$ and bring it to the canonical form with some complex eigenvalues $\lambda_I, I = 1, 2, 3, 4$. The value of the quartic invariant (4.2) will not change.

$$(x^{ab} + iy_{ab})_{\rm can} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_4 & 0 \end{pmatrix} \tag{4.9}$$

The relation between the complex coefficients λ_I, the parameters x^{ij} and y_{kl}, the matrix a_{ABD} and the black hole charges p^i and q_k [4] is given by the following dictionary:

$$\lambda_1 \;=\; x^{12} + iy_{12} \;=\; a_{111} + ia_{000} \;= -q_0 - ip^0$$

$$
\begin{aligned}
\lambda_2 &= x^{34}+iy_{34} = a_{001}+ia_{110} = -p^1+iq_1 \\
\lambda_3 &= x^{56}+iy_{56} = a_{010}+ia_{101} = -p^2+iq_2 \\
\lambda_4 &= x^{78}+iy_{78} = a_{100}+ia_{011} = p^3-iq_3
\end{aligned} \tag{4.10}
$$

If we now write the quartic $E_{7(7)}$ Cartan invariant in the canonical basis (x^{ij}, y_{ij}), $i, j = 1, ..., 8$:

$$
\begin{aligned}
I_4 \quad &= -(x^{12}y_{12}+x^{34}y_{34}+x^{56}y_{56}+x^{78}y_{78})^2 - 4(x^{12}x^{34}x^{56}x^{78}+y_{12}y_{34}y_{56}y_{78}) \\
&+4(x^{12}x^{34}y_{12}y_{34}+x^{12}x^{56}y_{12}y_{56}+x^{34}x^{56}y_{34}y_{56}+x^{12}x^{78}y_{12}y_{78}+x^{34}x^{78}y_{34}y_{78} \\
&+x^{56}x^{78}y_{56}y_{78})\ .
\end{aligned} \tag{4.11}
$$

then it may now be compared to Cayley's hyperdeterminant (1.1). We find

$$
I_4 = -\mathrm{Det}\ a \tag{4.12}
$$

The above discussion of $E_{7(7)}$ also applies, mutatis mutandis, to $E_7(C)$.

5 Decomposition of I_4

To understand better the entanglement we note that, as a result of (2.5), Cartan's invariant contains not one Cayley hyperdeterminant but seven! It may be written as the sum of seven terms each of which is invariant under $[SL(2)]^3$ plus cross terms. To see this, denote a 2 in one of the seven entries in (2.5) by A, B, C, D, E, F, G. So we may rewrite (2.5) as

$$
56 = (ABD)+(BCE)+(CDF)+(DEG)+(EFA)+(FGB)+(GAC) \tag{5.1}
$$

or symbolically

$$
56 = a+b+c+d+e+f+g \tag{5.2}
$$

Then I_4 is the singlet in $56\times 56\times 56\times 56$:

$$
\begin{aligned}
J_4 \sim\ & a^4+b^4+c^4+d^4+e^4+f^4+g^4+ \\
& 2[a^2b^2+b^2c^2+c^2d^2+d^2e^2+e^2f^2+f^2g^2+g^2a^2+ \\
& a^2c^2+b^2d^2+c^2e^2+d^2f^2+e^2g^2+f^2a^2+g^2b^2+ \\
& a^2d^2+b^2e^2+c^2f^2+d^2g^2+e^2a^2+f^2b^2+g^2c^2] \\
& +8[bcdf+cdeg+defa+efgb+fgac+gabd+abce]
\end{aligned} \tag{5.3}
$$

where products like

$$
\begin{aligned}
a^4 &= (ABD)(ABD)(ABD)(ABD) \\
&= \epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3}a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4}
\end{aligned} \tag{5.4}
$$

exclude four individuals (here Charlie, Emma, Fred and George), products like

$$
\begin{aligned}
a^2f^2 &= (ABD)(ABD)(FGB)(FGB) \\
&= \epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{F_3F_4}\epsilon^{G_3G_4}\epsilon^{D_2B_3}a_{A_1B_1D_1}a_{A_2B_2D_2}f_{F_3G_3B_3}f_{F_4G_4B_4}
\end{aligned} \tag{5.5}
$$

exclude two individuals (here Charlie and Emma), and products like

$$
\begin{aligned}
abce &= (ABD)(BCE)(CDF)(EFA) \\
&= \epsilon^{A_1A_4}\epsilon^{B_1B_2}\epsilon^{C_2C_3}\epsilon^{D_1D_3}\epsilon^{E_2E_4}\epsilon^{F_3F_4}a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4}
\end{aligned} \tag{5.6}
$$

exclude one individual (here George).

6 The black hole analogy

In the STU stringy black hole context [4, 5, 6, 7] the a_{ABC} are integers (corresponding to quantized charges) and hence the symmetry group is $[SL(2,Z)]^3$ rather than $[SL(2,C)]^3$. However, as discussed by Levay [8], there is a branch of quantum information theory which concerns itself with real qubits, called *rebits*, for which the a_{ABC} are real. (One difference remains, however: one may normalize the wave function, whereas for black holes there is no such restriction on the charges a_{ABC}.) It turns out that there are three reality classes which can be characterized by the hyperdeterminant

$$\begin{aligned} &1)\ \text{Det}\ a < 0 \\ &2)\ \text{Det}\ a = 0 \\ &3)\ \text{Det}\ a > 0 \end{aligned} \tag{6.1}$$

Case (1) corresponds to the non-separable or GHZ class [23], for example,

$$|\Psi\rangle = \frac{1}{2}(-|000\rangle + |011\rangle + |101\rangle + |110\rangle) \tag{6.2}$$

Case (2) corresponds to the separable (A-B-C, A-BC, B-CA, C-AB) and W classes, for example

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{6.3}$$

In the string/supergravity interpretation [4], cases (1) and (2) were shown to correspond to BPS black holes, for which half of the supersymmetry is preserved. Case (1) has non-zero horizon area and entropy ("large" black holes), and case (2) to vanishing horizon area and entropy ("small" black holes), at least at the semi-classical level. However, small black holes may acquire a non-zero entropy through higher order quantum effects. This entropy also has a quantum information interpretation involving bipartite entanglement of the three qubits [7].

Case (3) is also GHZ, for example the above GHZ state (6.2) with a sign flip

$$|\Psi\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \tag{6.4}$$

In the string/supergravity interpretation, case (3) corresponds to non-BPS black holes [7]. With four non-zero charges (q_0, p^1, p^2, p^3) in (4.10), for example, an extreme but non-BPS black hole [24] may be obtained by flipping the sign [25] of one of the charges. The canonical GHZ state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|111\rangle + \frac{1}{\sqrt{2}}|000\rangle \tag{6.5}$$

also belongs to case (3).

In the $N = 8$ theory, "large" and "small" black holes are classified by the sign of I_4:

$$\begin{aligned} &1)\ I_4 > 0 \\ &2)\ I_4 = 0 \end{aligned}$$

$$3)\ I_4 < 0 \tag{6.6}$$

Once again, non-zero I_4 corresponds to large black holes, which are BPS for $I_4 > 0$ and non-BPS for $I_4 < 0$, and vanishing I_4 to small black holes. However, in contrast to $N = 2$, case (1) requires that only 1/8 of the supersymmetry is preserved, while we may have 1/8, 1/4 or 1/2 for case (2).

It is worth noting that the charge orbits corresponding to non-zero I_4 are associated with the following cosets:

$$\frac{E_{7(7)}}{E_{6(2)}} \tag{6.7}$$

and

$$\frac{E_{7(7)}}{E_{6(6)}} \tag{6.8}$$

The large black hole solutions can be found [26] by solving the $N = 8$ classical attractor equations [22] when at the attractor value the Z_{AB} matrix, in normal form, becomes

$$Z_{AB} = \begin{pmatrix} Z\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.9}$$

for positive I_4 and

$$Z_{AB} = e^{i\pi/4}|Z| \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{6.10}$$

for negative I_4. These values exhibit the maximal compact symmetries $SU(6) \times SU(2)$ and $USp(8)$ for the positive and negative I_4, respectively.

If the phase in (4.6) vanishes (which is the case if the configuration preserves at least 1/4 supersymmetry [17]), I_4 becomes

$$I_4 = \lambda_1\lambda_2\lambda_3\lambda_4\ , \tag{6.11}$$

where we have defined λ_i by

$$\begin{aligned} \lambda_1 &= \rho_1 + \rho_2 + \rho_3 + \rho_4 \\ \lambda_2 &= \rho_1 + \rho_2 - \rho_3 - \rho_4 \\ \lambda_3 &= \rho_1 - \rho_2 + \rho_3 - \rho_4 \\ \lambda_4 &= \rho_1 - \rho_2 - \rho_3 + \rho_4 \end{aligned} \tag{6.12}$$

and we order the λ_i so that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq |\lambda_4|$. The charge orbits for the small black holes depend on the number of unbroken supersymmetries or the number of vanishing eigenvalues. The orbit is [14, 17, 27]

$$\frac{E_{7(7)}}{H_{1,2,3}} \tag{6.13}$$

where

$$H_1 = F_{4(4)} \ltimes T_{26} \qquad \lambda_1, \lambda_2, \lambda_3 \neq 0, \quad \lambda_4 = 0 \quad (1/8\ BPS)$$

$$H_2 = SO(5,6) \ltimes (T_{32} \times T_1) \qquad \lambda_1, \lambda_2 \neq 0, \quad \lambda_3, \lambda_4 = 0 \quad (1/4\ BPS)$$

$$H_3 = E_{6(6)} \ltimes T_{27} \qquad \lambda_1 \neq 0, \quad \lambda_2, \lambda_3, \lambda_4 = 0 \quad (1/2\ BPS) \tag{6.14}$$

For $N = 8$, as for $N = 2$, the large black holes correspond to the two classes of GHZ-type (entangled) states and small black holes to the separable or W class.

7 Subsectors

Having understood the analogy between $N = 8$ black holes and the tripartite entanglement of 7 qubits using $E_{7(7)}$, we may now find the analogy in the $N = 4$ case using $SL(2) \times SO(6,6)$ and the $N = 2$ case using $SL(2) \times SO(2,2)$.

For $N = 4$, as may be seen from (2.2), we still have an $[SL(2)]^7$ subgroup but now there are only 24 states

$$|\Psi\rangle = a_{ABD}|ABD\rangle + e_{EFA}|EFA\rangle + g_{GAC}|GAC\rangle \tag{7.1}$$

So only Alice talks to all the others. This is described by just those three lines passing through A in the Fano plane. Then the equations analagous to (5.1) and (5.2) are

$$(2,12) = (ABD) + (EFA) + (GAC) = a + e + g \tag{7.2}$$

and the corresponding quartic invariant, I_4, reduces to the singlet in $(2,12) \times (2,12) \times (2,12) \times (2,12)$.

$$I_4 \sim a^4 + e^4 + g^4 + 2[e^2g^2 + g^2a^2 + a^2e^2] \tag{7.3}$$

If we identify the 24 numbers $(a_{ABD}, e_{EFA}, g_{GAC})$ with (P^μ, Q_ν) with $\mu, \nu = 0, \dots 11$, this becomes [5, 18, 19]

$$I_4 = P^2Q^2 - (P.Q)^2 \tag{7.4}$$

which is manifestly invarinant under $SL(2) \times SO(6,6)$.

For $N = 2$, as may be seen from (2.2), we only an $[SL(2)]^3$ subgroup and there are only 8 states

$$|\Psi\rangle = a_{ABD}|ABD\rangle \tag{7.5}$$

This is described by just the ABD line in the Fano plane. This is simply the usual tripartite entanglement, for which

$$(2,2,2) = (ABD) = a \tag{7.6}$$

and the corresponding quartic invariant

$$I_4 \sim a^4 \tag{7.7}$$

is just Cayley's hyperdeterminant

$$I_4 = -\mathrm{Det}a \tag{7.8}$$

8 Conclusions

We note that the 56-dimensional Hilbert space given in (2.5) and (3.1) is not a subspace of the usual 2^7-dimensional seven-qubit Hilbert space given by $(2,2,2,2,2,2,2)$, but rather a direct sum of seven 2^3-dimensional three-qubit Hilbert spaces $(2,2,2)$. This is however, a subspace of the 3^7-dimensional seven-qutrit Hilbert space given by $(3,3,3,3,3,3,3)$. Under

$$[SL(3)]^7 \to [SL(2)]^7 \tag{8.1}$$

we have the decomposition

$$\begin{aligned}
&(3,3,3,3,3,3,3) \to \\
&1 \ \text{ term } \ \text{like } \ (2,2,2,2,2,2,2) \\
&7 \ \text{ terms } \ \text{like } \ (2,2,2,2,2,2,1) \\
&21 \ \text{ terms } \ \text{like } \ (2,2,2,2,2,1,1) \\
&35 \ \text{ terms } \ \text{like } \ (2,2,2,2,1,1,1) \\
&35 \ \text{ terms } \ \text{like } \ (2,2,2,1,1,1,1) \\
&21 \ \text{ terms } \ \text{like } \ (2,2,1,1,1,1,1) \\
&7 \ \text{ terms } \ \text{like } \ (2,1,1,1,1,1,1) \\
&1 \ \text{ term } \ \text{like } \ (1,1,1,1,1,1,1)
\end{aligned} \tag{8.2}$$

which contains

$$\begin{aligned}
&(2,2,1,2,1,1,1) \\
&+(1,2,2,1,2,1,1) \\
&+(1,1,1,2,2,1,2) \\
&+(2,1,1,1,2,2,1) \\
&+(1,2,1,1,1,2,2) \\
&+(2,1,2,1,1,1,2)
\end{aligned} \tag{8.3}$$

So the Fano plane entanglement we have described fits within conventional quantum information theory.

The Fano plane also finds application in switching networks that can connect any phone to any other phone. It is the 3-switching network for 7 numbers. However there also exists a 4-switching network for 13 numbers, a 5-switching network for 21 numbers, and generally an $(n+1)$-switching network for (n^2+n+1) numbers corresponding to the projective planes of order n [28]. It would be worthwhile pursuing the corresponding quantum bit entanglements.

Exceptional groups, such as $E_{7(7)}$, have featured in supergravity, string theory, M-theory and other speculative attempts at unification of the fundamental forces. However, it is unusual to find an exceptional group appearing in the context of qubit entanglement. It would be interesting to see whether it can be subject to experimental test.

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10 Note added

Just before the submission of version 2 of this paper to the arXiv, an interesting paper by Levay [29] appeared, which also describes the entanglement in terms of the Fano plane.

References

[1] A. Cayley, "On the theory of linear transformations," Camb. Math. J. 4 193-209,1845.

[2] A. Miyake and M. Wadati, "Multipartite entanglement and hyperdeterminants," [arXiv:quant-ph/0212146].

[3] V. Coffman, J. Kundu and W. Wooters, "Distributed entanglement," Phys. Rev. A61 (2000) 52306, [arXiv:quant-ph/9907047].

[4] M. J. Duff, "String triality, black hole entropy and Cayley's hyperdeterminant," arXiv:hep-th/0601134.

[5] M. J. Duff, J. T. Liu and J. Rahmfeld, "Four-dimensional string-string-string triality," Nucl. Phys. B **459**, 125 (1996) [arXiv:hep-th/9508094].

[6] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W. K. Wong, "STU black holes and string triality," Phys. Rev. D **54**, 6293 (1996) [arXiv:hep-th/9608059].

[7] R. Kallosh and A. Linde, "Strings, black holes, and quantum information," arXiv:hep-th/0602061.

[8] P. Levay, "Stringy black holes and the geometry of entanglement," Phys. Rev. D **74**, 024030 (2006) [arXiv:hep-th/0603136].

[9] M. J. Duff, "Hidden symmetries of the Nambu-Goto action," arXiv:hep-th/0602160.

[10] F. Klein, "Zur Theorie der Liniencomplexe des ersten und zweiten Grades", Math. Ann. 2, 198-226, 1870.

[11] E. Cartan, “Oeuvres completes”, (Editions du Centre National de la Recherche Scientifique, Paris, 1984).

[12] E. Cremmer and B. Julia, “The SO(8) Supergravity,” Nucl. Phys. B **159**, 141 (1979).

[13] R. Kallosh and B. Kol, “E(7) Symmetric Area of the Black Hole Horizon,” Phys. Rev. D **53**, 5344 (1996) [arXiv:hep-th/9602014].

[14] S. Ferrara and M. Gunaydin, “Orbits of exceptional groups, duality and BPS states in string theory,” Int. J. Mod. Phys. A **13**, 2075 (1998) [arXiv:hep-th/9708025].

[15] V. Balasubramanian, F. Larsen and R. G. Leigh, “Branes at angles and black holes,” Phys. Rev. D **57**, 3509 (1998) [arXiv:hep-th/9704143];

[16] M. Gunaydin, K. Koepsell and H. Nicolai, “Conformal and quasiconformal realizations of exceptional Lie groups,” Commun. Math. Phys. **221**, 57 (2001) [arXiv:hep-th/0008063].

[17] S. Ferrara and J. M. Maldacena, “Branes, central charges and U-duality invariant BPS conditions,” Class. Quant. Grav. **15**, 749 (1998) [arXiv:hep-th/9706097].

[18] M. Cvetic and D. Youm, “All the Static Spherically Symmetric Black Holes of Heterotic String on a Six Torus,” Nucl. Phys. B **472**, 249 (1996) [arXiv:hep-th/9512127].

[19] M. Cvetic and A. A. Tseytlin, “Solitonic strings and BPS saturated dyonic black holes,” Phys. Rev. D **53**, 5619 (1996) [Erratum-ibid. D **55**, 3907 (1997)] [arXiv:hep-th/9512031].

[20] I. Bena and P. Kraus, “Microscopic description of black rings in AdS/CFT,” JHEP **0412**, 070 (2004) [arXiv:hep-th/0408186]

[21] I. Bena, P. Kraus and N. P. Warner, “Black rings in Taub-NUT,” Phys. Rev. D **72**, 084019 (2005) [arXiv:hep-th/0504142].

[22] S. Ferrara, R. Kallosh and A. Strominger, “N=2 extremal black holes,” Phys. Rev. D **52**, 5412 (1995) [arXiv:hep-th/9508072].

[23] D. M. Greenberger, M. Horne and A. Zeilinger, in *Bell’s Theorem, Quantum Theory and Conceptions of the Universe*, ed. M. Kafatos (Kluwer, Dordrecht, 1989)

[24] M. J. Duff and J. Rahmfeld, “Massive string states as extreme black holes,” Phys. Lett. B **345**, 441 (1995) [arXiv:hep-th/9406105].

[25] M. J. Duff and J. Rahmfeld, “Bound States of Black Holes and Other P-branes,” Nucl. Phys. B **481**, 332 (1996) [arXiv:hep-th/9605085].

[26] S. Ferrara and R. Kallosh, “On N = 8 attractors,” Phys. Rev. D **73**, 125005 (2006) [arXiv:hep-th/0603247].

[27] H. Lu, C. N. Pope and K. S. Stelle, “Multiplet structures of BPS solitons,” Class. Quant. Grav. **15**, 537 (1998) [arXiv:hep-th/9708109].

[28] E. Pegg Jr, “The Fano Plane”, MAA online,
http://www.maa.org/editorial/mathgames/mathgames_05_30_06.html

[29] P. Levay, “Strings, black holes, the tripartite entanglement of seven qubits and the Fano plane,” arXiv:hep-th/0610314.